Boundary Element Method

MATH-0471

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Potential problem in Ω:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \bar{u} & \text{on } \Gamma_D \quad \text{(Dirichlet)} \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \bar{u}_n & \text{on } \Gamma_N \quad \text{(Neumann)} \end{cases}$$

where

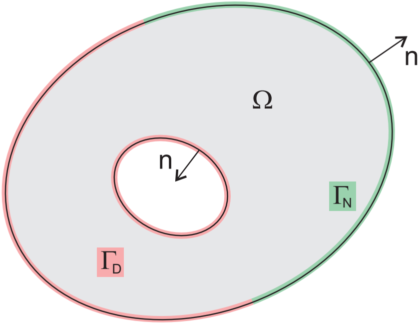
- u is the scalar unknown,
- Γ is the boundary of Ω and $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$,
- $\frac{\partial u}{\partial \boldsymbol{n}} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the outward normal to Ω.

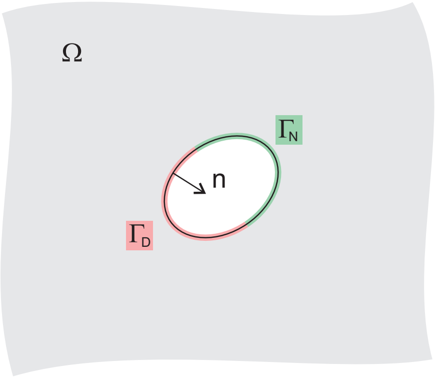
Note that the PDE is homogeneous but the boundary conditions are not.

We follow here the notations from the book of [Katsikadelis, 2016], which is freely available from the ULiege library website (use VPN).

Boundary Element Method

Potential problem





The domain can be finite (left) or infinite (right).

Boundary Element Method

Fundamental solution / Free space Green's function in 2D

If we ignore the boundary conditions, a **fundamental solution** v of the Laplace problem defined in $\mathbb{R}^2$ can be found by solving the PDE with a Dirac impulse as right-hand side.

$$\Delta v = \delta(\boldsymbol{Q} - \boldsymbol{P})$$

where $\boldsymbol{P} = \boldsymbol{P}(x, y)$ is the position of the Dirac impulse and $\boldsymbol{Q} = \boldsymbol{Q}(\xi, \eta)$ is the position where v is evaluated

We obtain

$$v(\boldsymbol{Q}, \boldsymbol{P}) = \frac{1}{2\pi} \ln r$$

with $r = \sqrt{(\xi - x)^2 + (\eta - y)^2} = ||\boldsymbol{Q} - \boldsymbol{P}||$

The fundamental solution is singular at $\boldsymbol{Q} = \boldsymbol{P}$.

Boundary Element Method

Representation formula in the domain Ω

Green's second identity (also called Green's reciprocal identity):

$$\int_\Omega (v\,\Delta u - u\,\Delta v\,)\,d\Omega = \int_\Gamma \left(v\frac{\partial u}{\partial \boldsymbol{n}} - u\frac{\partial v}{\partial \boldsymbol{n}} \right)\,ds$$

From this equality, if we choose v as the fundamental solution introduced earlier, we can write a **representation formula** for the solution u of the Laplace equation in Ω (boundary excluded):

$$u(\boldsymbol{P}) = -\int_\Gamma \left(v(\boldsymbol{P},\boldsymbol{q})\frac{\partial u(\boldsymbol{q})}{\partial \boldsymbol{n}_q} - u(\boldsymbol{q})\frac{\partial v(\boldsymbol{P},\boldsymbol{q})}{\partial \boldsymbol{n}_q} \right)\,ds_q$$

Important remark concerning the notations:

- A capital letter $(\boldsymbol{P},\boldsymbol{Q})$ represents a point inside the domain Ω,
- A lowercase letter $(\boldsymbol{p},\boldsymbol{q})$ represents a point on the boundary of Ω.

This formula can be used to compute u in Ω if we know both $u(\boldsymbol{q})$ and $\frac{\partial u(\boldsymbol{q})}{\partial \boldsymbol{n}_q}$ on the boundary (but we don't: we have either one or the other).

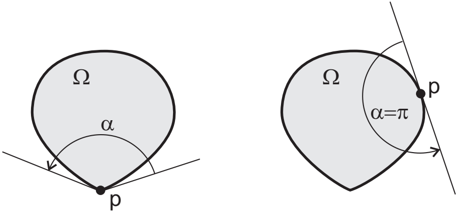
Boundary Element Method

Representation formula on the boundary Γ

From the Green's second identity, it is also possible to deduce a representation formula for the solution u lying on Γ:

$$\frac{\alpha}{2\pi} u(\boldsymbol{p}) = -\int_{\Gamma} \left(v(\boldsymbol{p}, \boldsymbol{q}) \frac{\partial u(\boldsymbol{q})}{\partial \boldsymbol{n}_q} - u(\boldsymbol{q}) \frac{\partial v(\boldsymbol{p}, \boldsymbol{q})}{\partial \boldsymbol{n}_q} \right) ds_q$$

where α is the angle of the corner made by the boundary at $\boldsymbol{p}$.



If the boundary is smooth, $\alpha = \pi$ and the coefficient in front of $u(\boldsymbol{p})$ is $\frac{1}{2}$.

This **boundary integral equation** can be used to calculate u and its derivative on the boundary.

Boundary Element Method

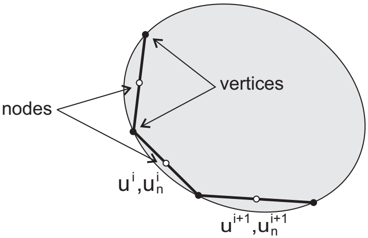
Principle of the BEM

- The Boundary Element Method consists in meshing the boundary of the problem with a series of N "boundary elements".
- On each element, the solution u is approximated by simple polynomials (constant, linear, etc.).
- Injecting this approximation into the boundary integral equation and taking into account the boundary conditions allows us to calculate the solution and its derivative on the whole boundary.
- Eventually, the solution inside the domain can be computed from the values on the boundary with the help of the representation formula.

Boundary Element Method

Constant boundary elements

- The simplest way to approximate the solution is to use a piecewise-linear approximation of the boundary (a series of oriented straight segments).
- On each element the solution u and its derivative u_n is considered constant (thus discontinuous across elements).
- We also define a node at the centre of each segment.



Boundary Element Method

Solving the boundary

Discretized Boundary Integral Equation:

$$\frac{1}{2}u_i = -\sum_{j=1}^{N}\int_{\Gamma_j} v(\boldsymbol{p}_i,\boldsymbol{q})\underbrace{\frac{\partial u(\boldsymbol{q})}{\partial \boldsymbol{n}_q}}_{u_n^j}\,ds_q + \sum_{j=1}^{N}\int_{\Gamma_j}\underbrace{u(\boldsymbol{q})}_{u^j}\frac{\partial v(\boldsymbol{p}_i,\boldsymbol{q})}{\partial \boldsymbol{n}_q}\,ds_q$$

Thanks to the node position centred in the middle of each segment, the boundary is locally smooth and $\alpha = \frac{1}{2}$.

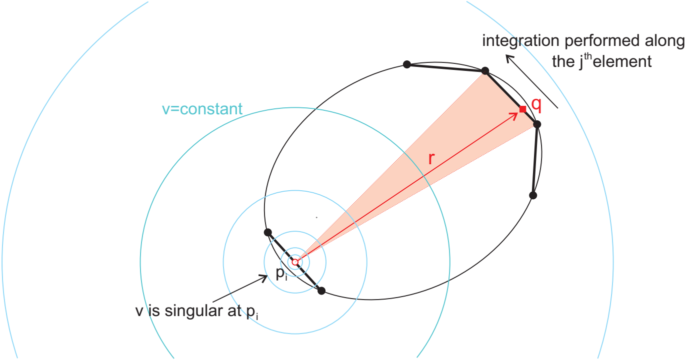
Reordering the terms:

$$-\frac{1}{2}u^i + \sum_{j=1}^{N}\underbrace{\left[\int_{\Gamma_j}\frac{\partial v(\boldsymbol{p}_i,\boldsymbol{q})}{\partial \boldsymbol{n}_q}\,ds_q\right]}_{\hat{H}_{ij}}u^j = \sum_{j=1}^{N}\underbrace{\left[\int_{\Gamma_j} v(\boldsymbol{p}_i,\boldsymbol{q})\,ds_q\right]}_{G_{ij}}u_n^j$$

Boundary Element Method

Solving the boundary

$$\sum_{j=1}^{N} H_{ij} u^j = \sum_{j=1}^{N} G_{ij} u_n^j \qquad \text{with} \quad \begin{cases} H_{ij} = \displaystyle\int_{\Gamma_j} \frac{\partial v(\boldsymbol{p}_i, \boldsymbol{q})}{\partial \boldsymbol{n}_q} \, ds_q - \frac{1}{2}\delta_{ij} \\ G_{ij} = \displaystyle\int_{\Gamma_j} v(\boldsymbol{p}_i, \boldsymbol{q}) \, ds_q \end{cases}$$



Difficulties are expected when $i = j$ (singular integrand).

Boundary Element Method

Solving the boundary

In matrix form

$$[H]\{u\} = [G]\{u_n\}$$

Equations can be sorted according to the type of boundary condition prescribed at the corresponding node (e.g. Dirichlet first, then Neumann):

$$\begin{bmatrix} [H_D] & [H_N] \end{bmatrix} \begin{Bmatrix} \{u\}_D \\ \{u\}_N \end{Bmatrix} = \begin{bmatrix} [G_D] & [G_N] \end{bmatrix} \begin{Bmatrix} \{u_n\}_D \\ \{u_n\}_N \end{Bmatrix}$$

Since $\{u\}_D = \{\bar{u}\}$ and $\{u_n\}_N = \{\bar{u}_n\}$, we can gather the unknowns in the left-hand side:

$$\underbrace{\begin{bmatrix} [H_N] & -[G_D] \end{bmatrix}}_{[A]} \underbrace{\begin{Bmatrix} \{u\}_N \\ \{u_n\}_D \end{Bmatrix}}_{\{x\}} = \underbrace{\begin{bmatrix} -[H_D] & [G_N] \end{bmatrix} \begin{Bmatrix} \{\bar{u}\} \\ \{\bar{u}_n\} \end{Bmatrix}}_{\{b\}}$$

This system is then solved using a linear solver.

Boundary Element Method

Calculation of $[G]$

- G_{ij} with $i \neq j$ (Gauss quadrature):

$$G_{ij} = \int_{\Gamma_j} v(\boldsymbol{p}_i, \boldsymbol{q})\, ds_q = \int_{-1}^{1} \frac{1}{2\pi} \ln(r(\xi))\, \frac{l_j}{2}\, d\xi = \frac{l_j}{4\pi} \sum_{k=1}^{N^{GP}} \ln(r(\xi_k))\, w_k$$

where l_j is the length of element j and N^{GP} is the number of Gauss points with are located at ξ_k with weight w_k.

- G_{ii} (analytical integration):

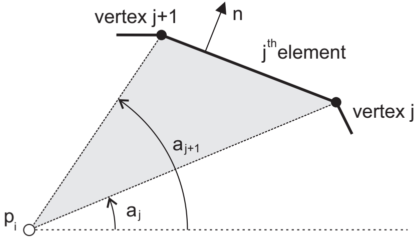
$$G_{ii} = \frac{l_i}{2\pi} \left(\ln \frac{l_i}{2} - 1 \right)$$

Boundary Element Method

Calculation of $[H]$

- H_{ij} with $i \neq j$ (analytical integration):

$$H_{ij} = \int_{\Gamma_j} \frac{\partial v(\boldsymbol{p}_i, \boldsymbol{q})}{\partial \boldsymbol{n}_q} \, ds_q = \frac{a_{j+1} - a_j}{2\pi}$$



- H_{ii} (analytical integration):

$$\hat{H}_{ii} = 0 \qquad \Rightarrow H_{ii} = -\frac{1}{2}$$

Boundary Element Method

Calculation of the solution inside Ω

From the representation formula:

$$u(\boldsymbol{P}) = -\int_{\Gamma} \left(v(\boldsymbol{P}, \boldsymbol{q}) \frac{\partial u(\boldsymbol{q})}{\partial \boldsymbol{n}_q} - u(\boldsymbol{q}) \frac{\partial v(\boldsymbol{P}, \boldsymbol{q})}{\partial \boldsymbol{n}_q} \right) ds_q$$

we can write:

$$u(\boldsymbol{P}) = -\sum_{j=1}^{N} \left[\int_{\Gamma_j} v(\boldsymbol{P}, \boldsymbol{q}) \, ds_q \right] u_n^j + \sum_{j=1}^{N} \left[\int_{\Gamma_j} \frac{\partial v(\boldsymbol{P}, \boldsymbol{q})}{\partial \boldsymbol{n}_q} \, ds_q \right] u^j$$

- The solution inside Ω requires the computation of boundary integrals with regular integrands (the singularity is located at $\boldsymbol{P}$). Refer to the expressions of H_{ij} and G_{ij} for $i \neq j$.
- If the solution at several points need to be computed, it can be easily performed in parallel.

Boundary Element Method

Summary

Advantages of the BEM:

- Only the surface of the problem should be meshed (the problem dimension is decreased by 1 compared to FEM).
- The domain can be infinite.
- The solution and its derivative can be easily evaluated anywhere in the domain.
- Works well for stationary, linear and homogeneous PDEs with non-homogeneous boundary conditions.

Typical applications:

- Potential problems: stationary heat conduction, electrostatics, irrotational flows, Darcy flows,
- Acoustics (Helmholtz equation),
- Slow viscous flows (Stokes equations),
- Elasticity (Lamé equations).

Boundary Element Method

Summary

Issues of the BEM:

- The method requires an **integral representation** of the solution and a fundamental solution to be calculated, which is sometimes impossible (e.g. nonlinear equations).
- Integrals of **singular functions** must be evaluated very precisely with special techniques for obtaining accurate results.
- The method relies on **non-sparse, non-symmetric matrices** which lead large memory requirement and CPU time. Thus, complicated acceleration techniques (fast multipole method, hierarchical matrices) are needed for 3D problems.

References

John T. Katsikadelis (2016)
The Boundary Element Method for Engineers and Scientists – Theory and Applications
Elsevier, *Second edition*, ISBN: 9780128044933