FEM for 2D Linear Elasticity

MATH-0471

January 27, 2022

FEM for 2D Linear Elasticity

Summary of equations

Mechanical equilibrium in Ω:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = 0 & \text{(translation)} \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^T & \text{(rotation)} \end{cases}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, ρ is the density and $\boldsymbol{b}$ represents the mechanical volumic forces.

and the **boundary conditions** on Γ, the boundary of Ω:

$$\begin{cases} \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } \Gamma_{\bar{\boldsymbol{u}}} & : \text{ prescribed displacement (Dirichlet)} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} & \text{on } \Gamma_{\bar{\boldsymbol{t}}} & : \text{ prescribed surface traction (Neumann)} \end{cases}$$

where $\boldsymbol{u}$ is the displacement, $\boldsymbol{n}$ is the outward unit normal to the boundary, $\Gamma_{\bar{\boldsymbol{u}}} \cup \Gamma_{\bar{\boldsymbol{t}}} = \Gamma$ and $\Gamma_{\bar{\boldsymbol{u}}} \cap \Gamma_{\bar{\boldsymbol{t}}} = \emptyset$

FEM for 2D Linear Elasticity

Summary of equations

Strain-displacement relationship (infinitesimal strains):

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right)$$

where small strains and small displacements are assumed.

Constitutive law (Hooke's law):

$$\boldsymbol{\sigma} = \mathbb{H} : \boldsymbol{\varepsilon}$$

with the 4^{th} order tensor

$$\mathbb{H}_{ijkl} = \frac{E}{2(1+\nu)}\delta_{ik}\delta_{jl} + \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} \quad \text{with } \begin{cases} E : \text{Young's modulus} \\ \nu : \text{Poisson's ratio} \end{cases}$$

for an isotropic material.

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Weak formulation

A **weak formulation** is obtained by multiplying the PDE by a function $\boldsymbol{w}(\boldsymbol{x})$ and by integrating over Ω:

$$\int_\Omega \boldsymbol{w} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \; dV + \int_\Omega \rho \boldsymbol{w} \cdot \boldsymbol{b} \, dV = 0$$

The first term of the left-hand side can be integrated by parts:

$$\int_\Omega \boldsymbol{w} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \; dV = \int_{\Gamma_{\bar{u}}} \boldsymbol{w} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, dS + \int_{\Gamma_{\bar{t}}} \boldsymbol{w} \cdot \underbrace{\boldsymbol{\sigma} \cdot \boldsymbol{n}}_{\bar{\boldsymbol{t}}} \, dS - \int_\Omega \boldsymbol{\nabla} \boldsymbol{w} : \boldsymbol{\sigma} \, dV$$

The integral on $\Gamma_{\bar{u}}$ can disappear if we choose test functions $\boldsymbol{w}(\boldsymbol{x})$ that vanish on this boundary.

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Weak formulation

Moreover, since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, we can write

$$\nabla \boldsymbol{w} : \boldsymbol{\sigma} = \underbrace{\tfrac{1}{2}\left(\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T\right)}_{\text{similar to } \boldsymbol{\varepsilon}(\boldsymbol{u})} : \boldsymbol{\sigma}$$

The weak formulation becomes[1]:

Weak formulation

Find $\boldsymbol{u}$ with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on $\Gamma_{\bar{u}}$ such that

$$\int_{\Gamma_{\bar{t}}} \boldsymbol{w} \cdot \bar{\boldsymbol{t}}\, dS - \int_{\Omega} \tfrac{1}{2}\left(\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T\right) : \boldsymbol{\sigma}\, dV + \int_{\Omega} \rho \boldsymbol{w} \cdot \boldsymbol{b}\, dV = 0$$

for all test functions $\boldsymbol{w}$ which are 0 on $\Gamma_{\bar{u}}$.

[1] This is also called "the principle of virtual work".
Notations in [Ponthot, 2020]: $\boldsymbol{w} \leftrightarrow \boldsymbol{\delta u}$ and $\delta\varepsilon \leftrightarrow \frac{1}{2}\left(\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T\right)$
and $\boldsymbol{u} + \boldsymbol{\delta u} = \bar{\boldsymbol{u}}$ on $\Gamma_{\bar{u}}$ ("kinematically admissible virtual displacement").

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Spatial discretization - Finite Element approximation

The discretization procedure is similar to the one followed for the Laplace/Poisson equation in MATH0024, except that the unknown field $\boldsymbol{u}(\boldsymbol{x}, t)$ is now a vector instead of a scalar. The equations are detailed here for the 2D case.

Using Cartesian coordinates, each component of $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{w}(\boldsymbol{x})$ are approximated using well-chosen shape functions N^k:

$$\begin{cases} u_1(\boldsymbol{x}, t) = \sum_k N^k(\boldsymbol{x})\, u_1^k(t) \\ u_2(\boldsymbol{x}, t) = \sum_k N^k(\boldsymbol{x})\, u_2^k(t) \end{cases} \quad \text{and} \quad \begin{cases} w_1(\boldsymbol{x}) = \sum_k N^k(\boldsymbol{x})\, w_1^k \\ w_2(\boldsymbol{x}) = \sum_k N^k(\boldsymbol{x})\, w_2^k \end{cases}$$

where u_i^k is the i^{th} coordinate of node k (also called “degree of freedom”).

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Spatial discretization - Finite Element approximation

In matrix form (assuming a mesh of n nodes in a 2D space):

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathbf{N}(\boldsymbol{x})\, \boldsymbol{d}(t)$$

with

$$\mathbf{N}(\boldsymbol{x}) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}$$

$$\boldsymbol{d}^T = \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 & \dots & u_1^n & u_2^n \end{bmatrix}$$

Similarly, for the test functions:

$$\boldsymbol{w}(\boldsymbol{x}) = \mathbf{N}(\boldsymbol{x})\, \boldsymbol{h}$$

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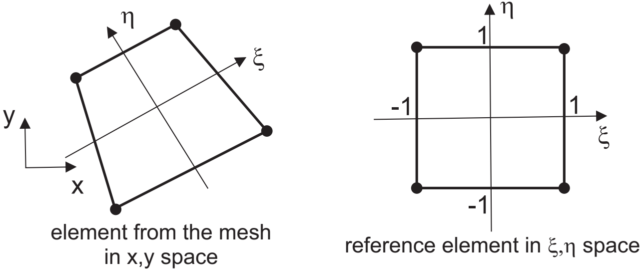
Isoparametric Finite Elements



Isoparametric finite elements (m nodes) are a classical choice. Their geometry is interpolated with the same shape functions[2] as the unknown field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the test functions $\boldsymbol{w}(\boldsymbol{x})$.

$$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{k=1}^{m} N^k(\boldsymbol{\xi})\,\boldsymbol{x}^k$$

where $\boldsymbol{\xi}$ are reduced coordinates (usually $\xi_i \in [-1, 1]$)

[2] N^k: superscript k is related nodes where $N^k = 1$

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3D Hooke's law (matrix form)

Voigt's notation: in a practical implementation, 4^{th} and 2^{nd}-order tensors are replaced by matrices and vectors respectively.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{bmatrix}$$

such that

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = \boldsymbol{\varepsilon}^{v} \cdot \boldsymbol{\sigma}^{v}$$

Hooke's law becomes:

$$\boldsymbol{\sigma}^{v} = \mathbf{H}\,\boldsymbol{\varepsilon}^{v}$$

Remark: $\boldsymbol{\varepsilon}^{v}$ involves $2\times\varepsilon_{ij} = \gamma_{ij}$ (shear strain/angle) for $i \neq j$.

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2D Hooke's law (matrix form)

A very common 2D assumption is the **plane-stress hypothesis** which is valid for thin structures ($\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$):

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} = \underbrace{\frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu}{2} \end{bmatrix}}_{\boldsymbol{H}^{-1}} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ \sigma_{12} \\ 0 \\ 0 \end{bmatrix}$$

The 2D Hooke's law is obtained by inverting this relationship:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\boldsymbol{H}_{2D}} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

This 3×3 matrix is used as $\boldsymbol{H}$ in the following equations.

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Strain-displacement relationship

Using Voigt's notation, the strain-displacement relationship becomes:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

or, symbolically

$$\boldsymbol{\varepsilon}^v = \boldsymbol{\partial}\, \boldsymbol{u}$$

Remarks:

- ε_{33} does not appear in the 2D equations. However, it can be computed, if needed, from the stresses:

$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)$$

- Strains (and stresses) are discontinuous across finite elements boundaries unless $\boldsymbol{u}(\boldsymbol{x})$ is C^1, which is almost never the case in practise.

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Strains/stresses as a function of the unknowns

Computation of **strains** from the nodal displacements $\boldsymbol{d}$:

$$\varepsilon^v = \boldsymbol{\partial}\, \boldsymbol{u} = \boldsymbol{\partial}\, \mathbf{N}\, \boldsymbol{d} = \mathbf{B}\, \boldsymbol{d}$$

with

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} N^1 & 0 & \dots & N^m & 0 \\ 0 & N^1 & \dots & 0 & N^m \end{bmatrix}$$

The **stresses** can also be obtained from the nodal displacement $\boldsymbol{d}$:

$$\boldsymbol{\sigma}^v = \mathbf{H}\, \mathbf{B}\, \boldsymbol{d}$$

FEM for 2D Linear Elasticity

Discretized weak formulation

Back to the **weak formulation**

$$\int_{\Gamma_{\bar{t}}} \boldsymbol{w} \cdot \bar{\boldsymbol{t}} \, dS - \int_\Omega \underbrace{\frac{1}{2}\left(\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^T\right) : \boldsymbol{\sigma}}_{(\mathbf{B}\boldsymbol{h})\cdot\boldsymbol{\sigma}^v} \, dV + \int_\Omega \rho \boldsymbol{w} \cdot \boldsymbol{b} \, dV = 0$$

Replacing $\boldsymbol{w} = \mathbf{N}\boldsymbol{h}$ and $\boldsymbol{u} = \mathbf{N}\boldsymbol{d}$ and $\boldsymbol{\sigma}^v = \mathbf{H}\,\mathbf{B}\,\boldsymbol{d}$

$$\boldsymbol{h}^T \left(\int_{\Gamma_{\bar{t}}} \mathbf{N}^T \bar{\boldsymbol{t}} \, dS \right) - \boldsymbol{h}^T \left(\int_\Omega \mathbf{B}^T \mathbf{H} \mathbf{B} \, dV \right) \boldsymbol{d} + \boldsymbol{h}^T \left(\int_\Omega \rho \mathbf{N}^T \boldsymbol{b} \, dV \right) = 0$$

This relationship should be satisfied for any $\boldsymbol{h}$.

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Discretized FE equations

This leads to the system of discretized equations:

$$\underbrace{\left(\int_{\Omega} \mathbf{B}^T \mathbf{H} \mathbf{B}\, dV\right)}_{\mathbf{K}} \boldsymbol{d} = \underbrace{\left(\int_{\Gamma_{\bar{\boldsymbol{t}}}} \mathbf{N}^T \bar{\boldsymbol{t}}\, dS\right) + \left(\int_{\Omega} \rho\, \mathbf{N}^T \boldsymbol{b}\, dV\right)}_{\boldsymbol{f}}$$

Discretized equations

$$\mathbf{K}\boldsymbol{d} = \boldsymbol{f}$$

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Finite Elements

Assembly:

- The integrals involved in the calculation of $\mathbf{K}$ and $\boldsymbol{f}$ are expressed as a sum of integrals over each finite element. These elemental contributions are then assembled (summed) in a large and structural matrix and vector.
- $\mathbf{K}$ is a symmetric sparse matrix and this property should be exploited to reduce the required storage and to efficiently solve the linear system of equations.

Boundary conditions:

- The equations of the system corresponding to nodes where Dirichlet boundary conditions are prescribed should be discarded and replaced by equations enforcing these conditions ($\boldsymbol{u} = \bar{\boldsymbol{u}}$) at these nodes.
- Homogeneous Dirichlet boundary conditions ($\boldsymbol{u} = 0$) lead to remove the lines and columns of the matrices corresponding to the components of the displacement where these conditions are prescribed.

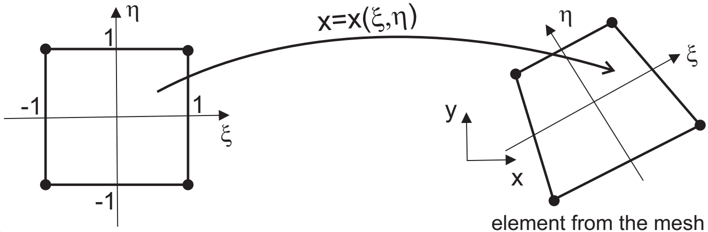
FEM for 2D Linear Elasticity

Practical calculation of the elemental stiffness matrices

$$K_{ij} = \int_{\Omega} B_{ki}(\boldsymbol{x})\, H_{kl}\, B_{lj}(\boldsymbol{x})\, dV$$

Isoparametric finite elements:

$$K_{ij} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B_{ki}(\boldsymbol{\xi})\, H_{kl}\, B_{lj}(\boldsymbol{\xi})\, \underbrace{\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)}_{\text{jacobian}}\, d\boldsymbol{\xi}$$



reference element in ξ,η space

element from the mesh in x,y space

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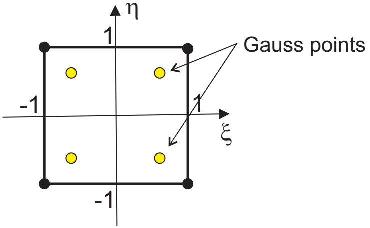
Gauss quadrature

This integral is calculated using a Gauss quadrature: (N^{GP} chosen $\rightarrow \boldsymbol{\xi}^p$, w_p)

$$K_{ij} \approx \sum_{p=1}^{N^{\mathrm{GP}}} \underbrace{B_{ki}(\boldsymbol{\xi})\, H_{kl}\, B_{lj}(\boldsymbol{\xi}) \det \mathbf{J}}_{\text{all the factors evaluated at } \boldsymbol{\xi}=\boldsymbol{\xi}^p} \; w_p$$

with N^{GP}, the number of Gauss points, $\boldsymbol{\xi}^p$ the positions and w_p, the weights.



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Jacobian of isoparametric elements at each Gauss points

Jacobian matrix[3]: let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$

$$J_{ij}(\boldsymbol{\xi}) = \frac{\partial x_i}{\partial \xi_j} = \frac{\partial N^k(\boldsymbol{\xi})}{\partial \xi_j} x_i^k$$

where x_i^k is the i^{th} coordinate of the k^{th} node of the finite element.

The Jacobian matrix and its determinant must be evaluated at each Gauss point $(\boldsymbol{\xi} = \boldsymbol{\xi}^p)$ of each finite element.

Note: the derivatives of the shape functions evaluated at the Gauss points $\frac{\partial N^k}{\partial \xi_j}(\boldsymbol{\xi}^p)$ are the same for all elements and can be computed once.

[3] transposed in [Ponthot, 2020]

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Derivatives of the shape functions

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} N^1 & 0 & \dots & N^m & 0 \\ 0 & N^1 & \dots & 0 & N^m \end{bmatrix}$$

The matrix $\mathbf{B}$ contains the derivatives of the shape functions with respect to $\boldsymbol{x}$ which should be transformed into derivatives of the shape functions with respect to $\boldsymbol{\xi}$ using the Jacobian matrix:

$$\frac{\partial}{\partial \xi_i} = \left(\frac{\partial x_j}{\partial \xi_i}\right) \frac{\partial}{\partial x_j} = J_{ji} \frac{\partial}{\partial x_j} \qquad \Rightarrow \boldsymbol{\nabla}_{\boldsymbol{x}} = \mathbf{J}^{-T} \boldsymbol{\nabla}_{\boldsymbol{\xi}}$$

Thus, the derivative of N^i with respect to $\boldsymbol{x}$ can be computed by this formula:

$$\boldsymbol{\nabla}_{\boldsymbol{x}} N^i = \mathbf{J}^{-T} \boldsymbol{\nabla}_{\boldsymbol{\xi}} N^i$$

References

J.-P. Ponthot (2020)
An Introduction to the Finite Element Method
Lecture notes, chapters 10 – 11.